Towards Understanding the Complexity of Fragments of Word Equations

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- Let $A := \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}$ be a set of terminal symbols.
- Let $U, V \in (X \cup A)^*$. Then $U = V$ is a word equation.
- Solutions are substitutions of terminal words for the variables such that the LHS and RHS become identical.
- In other words, solutions are terminal-preserving homomorphisms $h : (X \cup A)^* \to A^*$ such that $h(U) = h(V)$.

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Given a word equation $U = V$, does there exist a solution h? Does there exist a solution h satisfying some additional constraints?

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- From a theoreticians point of view, this is a natural tactic for improving understanding overall.
- Some fragments may be more relevant to practical applications than the general case anyway.
- We need tools for showing upper bounds in particular.

An Interesting Fragment

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- Satisfiability of quadratic equations remains NP-hard [Diekert, Robson '99].
- There is simple proof of decidability (via Nielson Transformations).
- As with the general case, inclusion in NP remains a long-standing open problem.

A Hard/Simple Fragment

Strictly Regular-Ordered Equations (SROWEs)

- SROWEs are equations $U = V$ which have the form

$$u_0 x_1 u_1 x_2 u_2 \dots x_n u_n = v_0 x_1 v_1 x_2 v_2 \dots x_n v_n.$$

where $u_i, v_i \in A^*$ and the x_is are (distinct) variables.

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- Inclusion in NP is straightforward: minimal solutions will be short (linear).
- Showing the lower bounds is much more involved, and is done by reduction from 3-Partition.

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- **DFA Constraints:** for each variable x, $h(x)$ must belong to the language of some DFA A_x.

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Theorem

The satisfiability problem for SROWEs with DFA, length, letter-counting and subword constraints is NP*-complete.*

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The Satisfiability Problem for ROWEs (without additional constraints) is NP*-complete.*

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- Moving toward more interesting/general classes, we need tools to reason about the non-minimality of solutions.
- We establish a condition for parts of a solution to be 'removable' (thus implying non-minimality) based on a representation of solutions as **chains of positions**.
- While this representation can be generalised to all equations, we shall see that it yields particular benefits for QWEs.

Chains Representation of Solutions to QWEs

Positions:

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$$x\ x\ \mathtt{a}\ \mathtt{a}\ y = z\ y\ \mathtt{b}\ z \to x_{(1)}x_{(2)}\mathtt{a}_{(1)}\mathtt{a}_{(2)}y_{(1)} = z_{(1)}y_{(2)}\mathtt{b}_{(1)}z_{(2)}$$

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- The set of **positions** w.r.t. (E, h) is

$$\mathcal{P}_E^h = \{(x, i, d) \mid x \in X \cup A \wedge 1 \leq |UV|_x \leq i \wedge 1 \leq d \leq |h(x)|\}$$

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- Hence there are $|h(U)| + |h(V)|$ total positions.
- Since h is a solution, every position has the same letter as its 'neighbour' on the other side of the equation.
- For any variable x and $i_1, i_2, d \in \mathbb{N}$, we must also have that the positions (x, i_1, d) and (x, i_2, d) have the same value.

Example:

$$x \ \ y \ \ \mathtt{a} \ \ y = \mathtt{a} \ \ z \ \ \mathtt{b}\,\mathtt{b} \ \ x \ \ z$$

$$h(x) = \mathtt{aaab}, \ h(y) = \mathtt{baa}, \ h(z) = \mathtt{aa}$$

<table>
<tr><td>U</td><td colspan="4">x</td><td colspan="3">y</td><td>a</td><td colspan="3">y</td></tr>
<tr><td>$h(U)$</td><td>a</td><td>a</td><td>a</td><td>b</td><td>b</td><td>a</td><td>a</td><td>a</td><td>b</td><td>a</td><td>a</td></tr>
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Example:

$$x_{(1)} \; y_{(1)} \; \mathtt{a}_{(1)} \; y_{(2)} = \mathtt{a}_{(2)} \; z_{(1)} \; \mathtt{b}_{(1)} \, \mathtt{b}_{(2)} \; x_{(2)} \; z_{(2)}$$

$$h(x) = \mathtt{aaab}, \; h(y) = \mathtt{baa}, \; h(z) = \mathtt{aa}$$

U	$x_{(1)}$				$y_{(1)}$			$\mathtt{a}_{(1)}$	$y_{(2)}$		
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$$h(x) = \mathtt{aaab},\; h(y) = \mathtt{baa},\; h(z) = \mathtt{aa}$$

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V	$\mathtt{a}_{(2)}$	$z_{(1)}$		$\mathtt{b}_{(1)}$	$\mathtt{b}_{(2)}$	$x_{(2)}$				$z_{(2)}$	

$(z, 2, 1)$

Chains Representation of Solutions to QWEs

Positions (Neighbour Relation):

- Every position has a unique **neighbour** corresponding to the same position on the other side of the equation.

Chains Representation of Solutions to QWEs

Example:

$$x_{(1)}\ y_{(1)}\ \mathtt{a}_{(1)}\ y_{(2)} = \mathtt{a}_{(2)}\ z_{(1)}\ \mathtt{b}_{(1)}\mathtt{b}_{(2)}\ x_{(2)}\ z_{(2)}$$

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$(x, 1, 3)$ and $(z, 1, 2)$ are **neighbours**

Chains Representation of Solutions to QWEs

Positions (Sibling Relation):

- Every position associated with a variable occurring twice has a **sibling** corresponding to the other occurrence of that variable.
- More formally, two positions (x, i, d) and (y, j, e) are siblings if $x = y$, $d = e$ and $i \neq j$.

Chains Representation of Solutions to QWEs

Example:

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V	$\mathtt{a}_{(2)}$	$z_{(1)}$		$\mathtt{b}_{(1)}$	$\mathtt{b}_{(2)}$	$x_{(2)}$				$z_{(2)}$	

$(x, 1, 3)$ and $(x, 2, 3)$ are **Siblings**

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- $p_1 = p$ and p_2 is the (unique) neighbour of p_1.

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- $p_1 = p$ and p_2 is the (unique) neighbour of p_1.
- for $i \geq 2$, if p_i corresponds to a terminal symbol or variable occurring only once, the chain terminates, and
- for $i \geq 2$, if p_i corresponds to a variable occurring twice, p_{i+1} is the neighbour of the sibling of p_i.

Construction of Chains (Example):

U	$x_{(1)}$				$y_{(1)}$			$\mathtt{a}_{(1)}$	$y_{(2)}$		
$h(U)$	a	a	a	b	b	a	a	a	b	a	a
$h(V)$	a	a	a	b	b	a	a	a	b	a	a
V	$\mathtt{a}_{(2)}$	$z_{(1)}$		$\mathtt{b}_{(1)}$	$\mathtt{b}_{(2)}$	$x_{(2)}$				$z_{(2)}$	

$(\mathrm{a}, 2, 1)$

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$h(U)$	a	a	a	b	b	a	a	a	b	a	a
$h(V)$	a	a	a	b	b	a	a	a	b	a	a
V	$a_{(2)}$	$z_{(1)}$		$b_{(1)}$	$b_{(2)}$	$x_{(2)}$				$z_{(2)}$	

$$(\mathrm{a}, 2, 1) \rightarrow (x, 1, 1) \rightarrow$$

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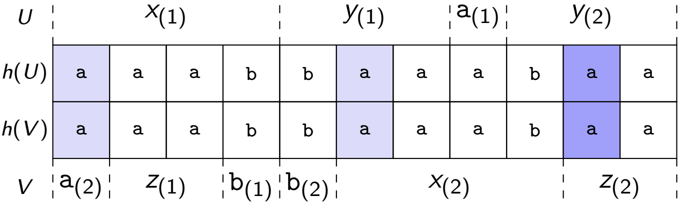
$$(\mathtt{a}, 2, 1) \to (x, 1, 1) \to (y, 1, 2) \to$$

Constructing the Chains

Example:



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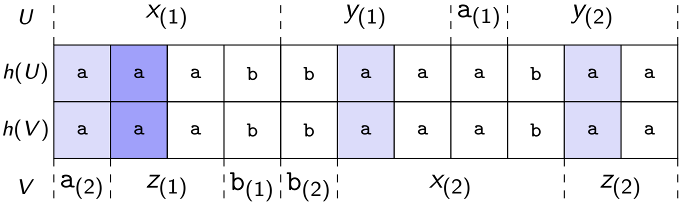

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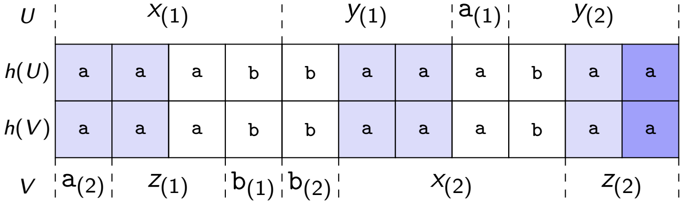

$$(\mathtt{a},2,1) \to (x,1,1) \to (y,1,2) \to (z,2,1) \to (x,1,2) \to (y,1,3) \to (z,2,2) \to$$

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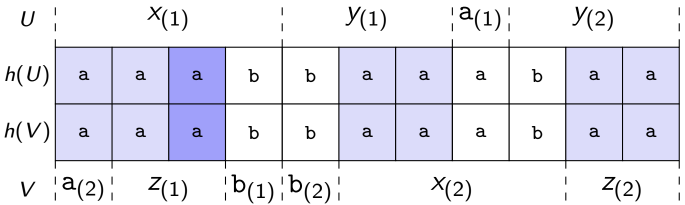

$$(\mathrm{a}, 2, 1) \rightarrow (x, 1, 1) \rightarrow (y, 1, 2) \rightarrow (z, 2, 1) \rightarrow (x, 1, 2) \rightarrow (y, 1, 3) \rightarrow (z, 2, 2) \rightarrow$$
$$\rightarrow (x, 1, 3)$$

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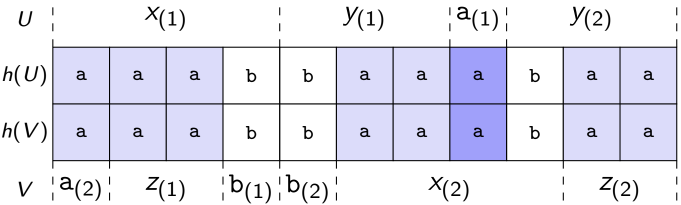

$$(\mathrm{a}, 2, 1) \to (x, 1, 1) \to (y, 1, 2) \to (z, 2, 1) \to (x, 1, 2) \to (y, 1, 3) \to (z, 2, 2) \to$$
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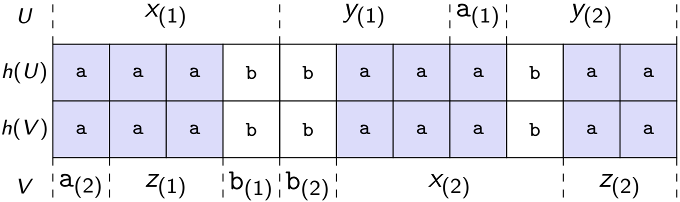

$$(a, 2, 1) \to (x, 1, 1) \to (y, 1, 2) \to (z, 2, 1) \to (x, 1, 2) \to (y, 1, 3) \to (z, 2, 2) \to$$
$$\to (x, 1, 3) \to (a, 1, 1)$$

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Example:

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$h(V)$	a	a	a	b	b	a	a	a	b	a	a
V	$\mathtt{a}_{(2)}$	$z_{(1)}$		$\mathtt{b}_{(1)}$	$\mathtt{b}_{(2)}$	$x_{(2)}$				$z_{(2)}$	

$$(\mathtt{a},2,1) \to (x,1,1) \to (y,1,2) \to (z,2,1) \to (x,1,2) \to (y,1,3) \to (z,2,2) \to$$

$$\to (x,1,3) \to (\mathtt{a},1,1)$$

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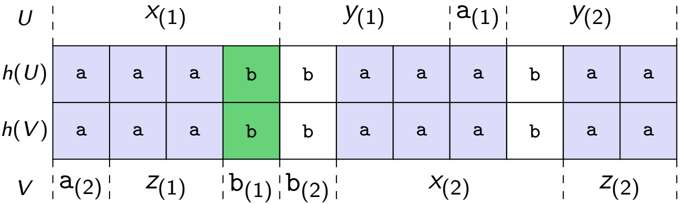
Constructing the Chains

Example:



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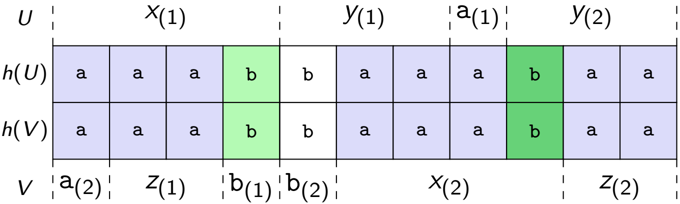

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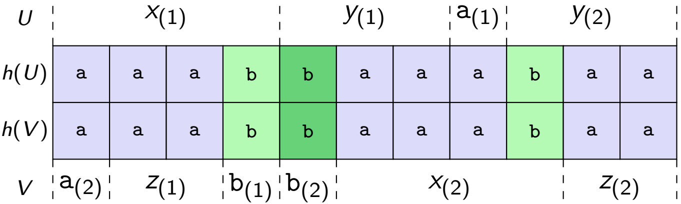

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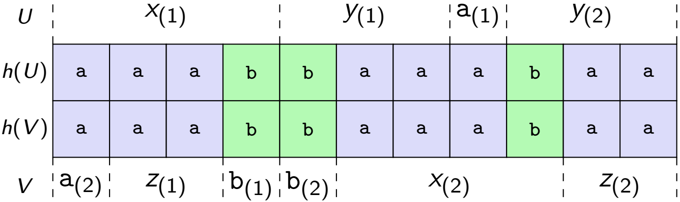

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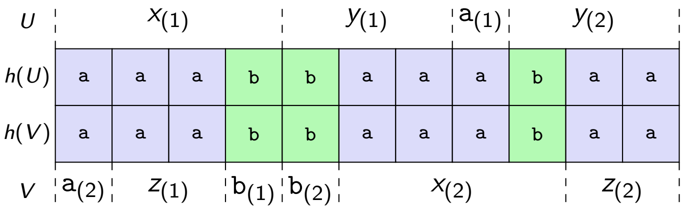

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Lemma

Let h be a minimal solution to some QWE $U = V$. Let $\mathcal{C}$ be the longest chain of h w.r.t $U = V$. Then $|h(U)| \leq |\mathcal{C}||UV|$.

From Chains to Words

Chain-Words:

- Let Γ be an alphabet of size $2|A \cup X|$ and let $\varphi : \mathcal{P}_E^h \to \Gamma$ be an a mapping such that $\varphi((x, i, d)) = \varphi((y, j, e))$ if and only if $x = y$ and $i = j$.

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Squares:

- A word u is a **square** if it is a direct repetition (it has the form $u = vv$ for some non-empty word v).

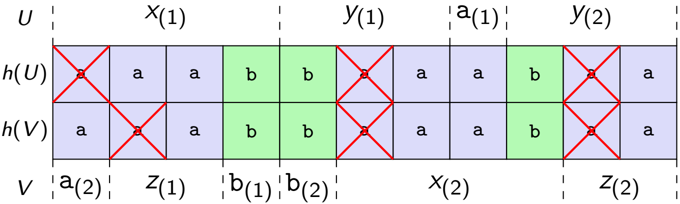
The Squares Lemma

Lemma (Squares Lemma)

Let E be a QWE, h be a solution to E and let w be a chain word of h w.r.t. E. If w contains a square, then h is not minimal.

The Squares Lemma

Example:



$$(\mathtt{a}, 2, \not{1}) \to (\mathbf{x}, \mathbf{1}, \not{1}) \to (\mathbf{y}, \mathbf{1}, \not{2}) \to (\mathbf{z}, \mathbf{2}, \not{1}) \to (\mathbf{x}, \mathbf{1}, \not{2}) \to (\mathbf{y}, \mathbf{1}, \not{3}) \to (\mathbf{z}, \mathbf{2}, \not{2}) \to$$

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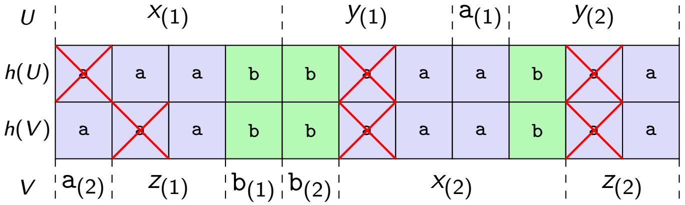

$h'(x) = \mathtt{aab}, h'(y) = \mathtt{ba}, h'(z) = \mathtt{a}$ is also a solution!

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- We can also generalise it to work with additional constraints on solutions such as regular constraints, and involutions.
- The existence of a square in one of the chain-words corresponds to a set of factors in the solution which may be 'pumped'.
- Unfortunately, proving that long solutions/chain-words must contain squares seems very difficult.

Upper Bounds

ROWEs

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- Thus if a chain visits the same variable more than once, it must be consecutively. This would induce a "square", so by our lemma, the solution would not be minimal.
- In a minimal solution, each chain has length linear in the length of the equation.

Thus any minimal solution is quadratic in the length of the equation.

Upper Bounds

Variable Sparse QWEs (VSQWEs)

- We say that a QWE $U = V$ is **variable-sparse** if

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Regular-Reversed Word Equations (RRWEs)

- We say that a QWE $U = V$ is **regular-reversed** if it has the form:

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where $u_i, v_i \in A^*$ and the x_is are (distinct) variables.

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- The proof in this case requires a much more involved analysis, but relies mostly on the squares lemma.

Towards All QWEs

Open Problem

Does there exist an exponential(ish) function f such that, for any QWE $U = V$, if h is a solution and $|h(U)| > f(|UV|)$, then at least one of the chain-words of h w.r.t E contains a square?

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- So the question is, what does the set of all chain words of QWEs look like?
- We have a characterisation for **regular** equations (each variable occurs at most once per side).

Towards All QWEs

Theorem

Let w be a word and let Γ be the alphabet of letters occurring in w. There exists a regular word equation E with solution h such that w is a chain-word of h w.r.t. E if and only if there exist letters $\$, \# \notin \Gamma$ and linear orders $<_1$, $<_2$ on the sets $\Gamma \cup \{\#\}$ and $\Gamma \cup \{\$\}$ respectively such that for every $u \in \Gamma^*$ and $A, B, C, D \in \Gamma \cup \{\$, \#\}$ with $A \neq B$ and $C \neq D$, if AuC and BuD are both factors of $\#w\$$, then either that $A <_2 B$ and $C <_1 D$ or that $B <_2 A$ and $D <_1 C$.

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Corollary

Let E be a regular word equation and let h be a solution to E. Let w be a chain-word of h w.r.t. E. Let A, B, C, D be letters from w such that $A \neq B$ and $C \neq D$ Then for any word u, at least one of AuC, BuC, AuD, BuD is not a factor of w.

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Corollary

Let E be a regular word equation and let h be a solution to E. Let w be a chain-word of h w.r.t. E. Let n be the number of variables in E. Then w contains at most $2n - 1$ distinct factors of length 2.

Towards All QWEs

Lemma

Let w be a chain-word of some solution h w.r.t. some regular word equation E. Suppose that w contains a factor of the form $x_1x_2x_3x_4x_2x_1x_3$ such that x_3 is not a prefix of x_1 or x_2. Then some chain-word w' of h w.r.t. E contains a square, and h is not minimal.

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- It is possible to produce other patterns with prefix/suffix restrictions for which the lemma holds.

Questions?

Thank you!